Multi-unit Auctions with Budget Limits

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Abstract

We study multi-unit auctions where the bidders have a budget constraint, a situation very common in practice that has received relatively little attention in the auction theory literature. Our main result is an impossibility: there is no deterministic auction that (1) is individually-rational and dominant-strategy incentive-compatible, (2) makes no positive transfers, and (3) always produces a Pareto-optimal outcome. In contrast, we show that Ausubel’s “clinching auction” satisfies all these properties when the budgets are public knowledge. Moreover, we prove that the “clinching auction” is the unique auction that satisfies all these properties when there are two players. This uniqueness result is the cornerstone of the impossibility result. Few additional related results are given, including some results on the revenue of the clinching auction and on the case where the items are divisible.

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1 Introduction

The starting point of almost all of auction theory is the set of players’ valuations: how much value (measured in some currency unit) does each of them assigns to each possible outcome of the auction. When attempting actual implementations of auctions, a mismatch between theory and practice emerges immediately: budgets. Players often have a maximum upper bound on their possible payment to the auction – their budget.¹ Budgets are central elements in most of economic theory, but relatively little attention has been paid to them in auction theory. A concrete example is Google’s and Yahoo’s ad-auctions, where budgets are an important part of a user’s bid, and are perhaps even more real for the users than the rather abstract notion of a valuation.² Addressing budgets properly breaks down the usual quasi-linear setting, and because of this the VCG mechanism loses its incentive-compatibility. The design of dominant-strategy incentive-compatible mechanisms becomes significantly more involved.

The few relatively recent works that study auctions with budgets focus on several different directions. A first branch of works (Che and Gale, 1998; Benoit and Krishna, 2001) analyzes how budgets change the classic results on “standard” auction formats, showing for example that first-price auctions raise more revenue than second-price auctions when bidders are budget-constrained, and that the revenue of a sequential auction is higher than the revenue of a simultaneous ascending auction. A second branch of works (Laffont and Robert, 1996; Pai and Vohra, 2008) constructs single-item auctions that maximize the seller’s revenue, and a third branch (Maskin, 2000) considers the problem of “constrained efficiency”: maximizing the expected social welfare under Bayesian incentive compatibility constraints. A fourth branch (Borgs et al., 2005; Abrams, 2006), taken by the computer science community, tries to design dominant-strategy incentive-compatible multi-unit auctions that approximate the optimal revenue.

Our model in this paper is simple: There are $m$ identical indivisible units for sale, and each bidder $i$ has a private value $v_i$ for each unit, as well as a budget limit $b_i$ on the total amount he may pay. We also consider the limiting case where $m$ is large by looking at auctions of a single infinitely-divisible good. Our assumption is that bidders are utility-maximizers, where $i$’s utility from acquiring $x_i$ units (or a fraction of $x_i$ of the good, in the infinitely divisible good case) and paying $p_i$ is $u_i = x_i \cdot v_i - p_i$, as long as the price is within budget, $p_i \leq b_i$, and is negative infinity (infeasible) if $p_i > b_i$.³ Thus the utility is linear in the payment only for outcomes in which the payment is at most the budget. This makes our setting non-quasi-linear.

We study the fundamental question of how to produce efficient allocations in an incentive-compatible way, using the most basic solution concept of dominant-strategies. As the setting is not quasi-linear, allocational efficiency is not uniquely defined since different outcomes are preferred by different players⁴. We thus focus at a weak efficiency requirement: Pareto-optimality, i.e.,

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¹ The nature of what this budget limit means for the bidders themselves is somewhat of a mystery since it often does not seem to simply reflect the true liquidity constraints of the bidding firm. There seems to be some risk control element to it, some purely administrative element to it, some bounded-rationality element to it, and more.

² See the paper of Nisan et al. (2009) for a more detailed discussion on Google’s auction structure.

³ This model naturally generalizes to any type of multi-item auction: bidders have a valuation $v_i(\cdot)$ and a budget $b_i$, and their utility from acquiring a set $S$ of items and paying $p_i$ for them is $u_i(S) = v_i(S) - p_i$, as long as $p_i \leq b_i$, and negative infinity if the budget has been exceeded $p_i > b_i$. It is interesting to note that the “demand-oracle model” (see e.g. Blumrosen and Nisan (2007)) represents such bidders as well. Analyzing combinatorial auctions with budget limits, even in simple settings such as additive valuations, is clearly a direction for future research.

⁴ In quasi-linear settings any Pareto-optimal outcome must optimize the “social-welfare” – the sum of bidders valuations – and thus efficiency is justifiably interpreted as maximizing social-welfare.
outcomes where it is impossible to strictly improve the utility of some players without hurting those of others. There exist many Pareto-optimal allocation rules, and we wish to identify those that are implementable in dominant-strategies. Following standard terminology in computer science, a dominant-strategy incentive-compatible mechanism is called a truthful mechanism throughout. Thus, we ask what are the truthful mechanisms whose outcomes are always Pareto-optimal in our setting.

**Main results.** Our main result is an impossibility: *there is no deterministic, truthful, and Pareto-optimal auction*, for any finite number $m > 1$ of units of an indivisible good and any $n \geq 2$ number of players.\(^5\) The cornerstone of the analysis is a characterization result for the case where budgets are public information. For this case we show that Ausubel’s “clinching auction” (Ausubel, 2004) is Pareto-optimal and truthful.\(^6\) Moreover we show that the clinching auction is the *unique* (up to tie-breaking) auction that satisfies the above properties, when there are exactly two bidders. The assumption of public budgets was made many times before us, e.g. by Laffont and Robert (1996) and in Maskin (2000), and thus we do not wish to argue that private budgets are more plausible than public budgets. On the contrary, we view the second result as a useful *positive* result, which completely pin-points the (only) possible truthful mechanism that is also Pareto-optimal.

We emphasize that the main point of the uniqueness result is *not* that payments are unique for the allocation rule of the clinching auction. Indeed this would easily follow from the Revenue Equivalence Theorem (which in turn follows from the Envelope Theorem, as shown in Milgrom and Segal (2002)). Rather, the main point is that the allocation rule itself is the *unique allocation rule* that satisfies the above properties, most notably truthfulness and Pareto-optimality. In contrast to the quasi-linear setting, were welfare-maximization is the only Pareto-optimal rule (regardless of the other properties), in our setting there exist many deterministic allocation rules that are Pareto-optimal, individually-rational, and with no-positive-transfers. There also exist many deterministic dominant-strategy mechanisms that are individually-rational and with no-positive-transfers (even with private budgets). We show that it is the combination of truthfulness and Pareto-optimality that yields the impossibility for private budgets, and the uniqueness for public budgets. This is quite surprising since a-priori there is no indication that these two properties clash.

Our characterization sheds light on the type of effects that budget limitations create. Recall that Ausubel’s auction gradually increases a price parameter, and bidders keep decreasing their demands for items at this price. Whenever the combined demand of the other bidders decreases strictly below available supply, bidder $i$ “clinches” the remaining quantity at the current price. Thus different amounts of units are acquired by bidders at different prices, and the total payment of a bidder is the sum of the prices of all units that he clinched throughout the auction. Ausubel shows that, in the quasi-linear setting, this auction yields exactly the VCG outcome and is thus truthful. The key property for truthfulness is that the demands for future items are fixed and independent of the prices at which previous items were acquired. With budgets, this property no longer holds, and demand for future items changes as a function of the remaining budget. If bidder $A$ slightly delays to report a demand decrease, bidder $B$ will pay as a result a slightly higher price for his acquired items, which reduces his future demand. In turn, the fact that bidder $B$ now has

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\(^5\)This theorem assumes “individual rationality” and “no positive transfers”, i.e. that bidders are not paid by the auction nor do they pay more than their value or budget. Without this, the budget limits can be easily side-stepped, e.g., by using a VCG mechanism that pays losers the total value of the others.

\(^6\)The original paper (Ausubel, 2004) makes several initial observations regarding the potential usefulness of the clinching auction when players have budgets, e.g. in the last paragraph of p. 1457 and in footnote 8.
a lower demand implies that bidder A pays a lower price for future items, and the contradiction to truthfulness becomes evident. Thus with private budgets this auction is no longer truthful, and our analysis implies that this difficulty is inherent to all Pareto-optimal allocation schemes. This seems to be the most common strategic problem that budgets introduce, see e.g. Benoit and Krishna (2001) and Brusco and Lopomo (2008). With public budgets (and private values), on the other hand, this manipulation is not possible. Moreover, for two bidders, the clinching auction is the unique truthful and Pareto-optimal auction.

To complete the picture we also analyze some revenue properties of the clinching auction in our setting with budgets. We show that, as the number of items increases and the “dominance” of each bidder decreases, the revenue of this mechanism approaches the revenue of a non-discriminatory monopoly, that knows the values and budgets of the players and determines a single unit-price in order to maximize revenue.

While in the quasi-linear setting, exact formulas for the outcome of the auction can be described (this is essentially the VCG mechanism), in our setting it is quite hard to come up with a parallel closed-form solution, especially in the infinitely-divisible good case for which the auction is a continuous time process. (This once again demonstrates the relative flexibility of ascending auctions versus direct mechanisms when one slightly changes the model). Nevertheless we present exact closed-form descriptions for an infinitely-divisible item and two players. These were certainly surprising for us, as they do not seem to resemble any previously considered auction format. In all cases, once the exact form is found, it is a straight forward exercise to verify truthfulness and Pareto-optimality. For example, if both players have equal budgets, i.e. w.l.o.g \( b_1 = b_2 = 1 \) and \( v_1 \leq v_2 \), then if \( \min(v_1, v_2) \leq 1 \) then the high-value player gets everything and pays the second highest value, and otherwise, the low-value player gets \( 1/2 - 1/(2 \cdot v_1^2) \) and pays \( 1 - 1/v_1 \) and the high-value player gets \( 1/2 + 1/(2 \cdot v_1^2) \) and pays 1. This unfamiliar format has of-course an underlying reasoning that we explain in the body of the paper. In parallel to the indivisible case, we show for the divisible case as well that when budgets are public, this auction is the unique anonymous Pareto-optimal and truthful deterministic auction. In a follow-up to our work, Bhattacharya, Conitzer, Munagala and Xia (2010) further analyze the divisible case, showing additional interesting properties. For example, if budgets are private, then the only profitable manipulation is to over-state one’s budgets.

As a last note, we point out that the impossibility for private budgets crucially depends on the assumption that players demand multiple items. Indeed, recently we have seen several works that present positive results for unit-demand players with budgets. For example, Aggarwal, Muthukrishnan, Pal and Pal (2009) show that an extension of the Demange-Gale-Sotomayor ascending auction is truthful and Pareto-optimal. Hatfield and Milgrom (2005) study a more abstract unit-demand model for players with non-quasi-linear utilities that generalizes both the Gale-Shapley stable-matching algorithm as well as the Demange-Gale-Sotomayor ascending auction, showing truthfulness and (in the context of our setting) Pareto-optimality. Ashlagi, Braverman, Hassidim, Lavi and Tennenholtz (2010) extend the generalized English auction to settings with budget-constraints, again showing truthfulness and Pareto-optimality.

The rest of the paper is organized as follows. We start with basic definitions and preliminary propositions in Section 2. The clinching auction (adjusted for our setting) is defined in Section 3, where we also analyze its basic properties: Pareto-optimality and truthfulness. Section 4 shows the uniqueness of this auction. Relying on this, Section 5 then proves the impossibility result for private budgets. Section 6 discusses some properties of the revenue of the clinching auction for
players with budgets, and Section 7 describes the closed-form mechanism for a divisible item.

2 Preliminaries and Notation

2.1 Outcomes

We will be considering auctions of $m$ identical indivisible items as well as the limiting case of a single infinitely divisible good.

We have $n$ bidders, where each bidder $i$ has a value $v_i$ for each unit he gets, and has a budget limit $b_i$ on his payment. Rather than explicitly declaring a bidder’s utility of going over-budget to be negative infinity, we will equivalently directly declare such cases to be infeasible.

Definition 2.1 An outcome $(x, p)$ is a vector of allocated quantities $x_1, ..., x_n$ and a vector of payments $p_1, ..., p_n$ with the following properties:

1. (Feasibility) In the case of finite $m$, $x_i$ must be a non-negative integer and $\sum_i x_i \leq m$. In the case of an infinitely divisible good, $x_i$ must be non-negative real and $\sum_i x_i \leq 1$.

2. (No Positive Transfers) $\sum_i p_i \geq 0$.

3. (Individual Rationality) $p_i \leq x_i \cdot v_i$.

4. (Budget Limit) $p_i \leq b_i$.

Our “no positive transfers” property is weak, in the sense that it allows the outcome to hand in payments to players. The only restriction is that, overall, the auctioneer does not hand money to the players. All our auctions satisfy the stronger version of the “no positive transfers” property, where for every player $i$ we have $p_i \geq 0$, i.e., no player gets money from the auction.\(^7\) The fact that we require the quantities $x_1, ..., x_n$ to be integers also implies that we require deterministic outcomes.

2.2 Auctions and Incentives

We will be formally considering only direct revelation auctions where bidders submit their value and budget to the auction, that based on the types $v_1, ..., v_n$ and $b_1, ..., b_n$ calculates the outcome $x_1, ..., x_n$ and $p_1, ..., p_n$. Our auctions have a very natural interpretation as dynamic ascending auctions\(^8\), but for simplicity we will just consider the auction mechanism as a black-box direct-revelation one.

Definition 2.2 A mechanism is truthful if for every $v = (v_1, ..., v_n)$, $b = (b_1, ..., b_n)$, and every possible manipulation $v'_i$ and $b'_i$, we have that $u_i = x_i \cdot v_i - p_i \geq x'_i \cdot v_i - p'_i = u'_i$, where $(x_i, p_i)$ are

\(^7\)The weak version is necessary for the uniqueness result. Consider, for example, the following mechanism for one item and two players with infinite budgets: the item is allocated to player 1 if $v_1 > 0$, and otherwise to player 2. No payments are made. One can verify that this is truthful. It is also Pareto-optimal if one requires the strong NPT property, since if $v_2 > v_1 > 0$, the only outcome that Pareto-dominates the one chosen by the mechanism is an outcome in which player 1 receives a payment of $v_1$, and player 2 receives the item and pays $v_1$. The sum of payments here is 0, so with weak NPT the outcome is not Pareto-optimal, and the mechanism can be ruled out.

\(^8\)As usual, the solution concept for the iterative version is ex-post-Nash.
the allocation and payment of $i$ for types $(v, b)$ and $(x_i', p_i')$ are the allocation and payment of $i$ for types $((v_i, v_i'), (b_i, b_i'))$.

A mechanism is truthful for the case of publicly known budgets if the definition above holds for all $v_i'$, having fixed $b_i' = b_i$.

### 2.3 Pareto-optimality

We start with the classic notion of Pareto optimality:

**Definition 2.3** An outcome $\{(x_i, p_i)\}$ is Pareto-optimal if for no other outcome $\{(x_i', p_i')\}$ are all players better off, $x_i'v_i - p_i' \geq x_i v_i - p_i$, including the auctioneer $\sum_i p_i' \geq \sum_i p_i$, with at least one of the inequalities strict.

Recall that an outcome requires by definition that payments will not exceed budgets, hence a player’s utility in some outcome $(x, p)$ is $x_i v_i - p_i$. The definition of an outcome also requires this utility to be non-negative.

In our setting, the notion of Pareto optimality is equivalent to a “no trade” condition that is much easier to work with. It essentially states that no money is “left on the table”, in the sense that no player can re-sell the items he received and make a profit:

**Proposition 2.4** An outcome $\{(x_i, p_i)\}$ is Pareto-optimal in the infinitely divisible case if and only if (a) $\sum_i x_i = 1$, i.e. the good is completely sold, and (b) for all $i$ such that $x_i > 0$ we have that for all $j$ with $v_j > v_i$, $p_j = b_j$. I.e. a player may get a non-zero outcome only if all higher value players have exhausted their budget.

For example, the outcome that awards all items to a buyer with highest value, and requires no payment, is Pareto-optimal, and indeed the two requirements of the claim hold (the second requirement holds in an empty way). The proof is given in Appendix A. A similar “no trade” property is equivalent to Pareto-optimality also in the case of finite $m$ (the proof is similar to the proof of the previous claim):

**Proposition 2.5** An outcome $\{(x_i, p_i)\}$ is Pareto-optimal in the case of finite $m$ if and only if (a) $\sum_i x_i = m$, i.e., all the units are sold, and (b) for all $i$ such that $x_i > 0$ we have that for all $j$ with $v_j > v_i$, $p_j > b_j - v_i$. I.e. a player may get a non-zero outcome only if there is no player with higher value that has larger remaining budget.

### 2.4 Warmup: The Proportional Share Auction

Recall that our main goal is to show the impossibility of constructing a mechanism that is Pareto-optimal and truthful when budgets are private. Before that, we wish to point out that the source of this difficulty is the fact that values and budgets may be very close to one another. If values are guaranteed to be sufficiently large with respect to the budgets then a simple mechanism exists:

**Definition 2.6** The proportional share auction for an infinitely divisible good allocates to each bidder $i$ a fraction $x_i = b_i / \sum_j b_j$ of the good and charges him his total budget $p_i = b_i$. 
Proposition 2.7 Let $\alpha_i = b_i / \sum_j b_j$ be the budget share of player $i$. The proportional-share auction with $x_i = b_i / \sum_j b_j$ and $p_i = b_i$ is Pareto Optimal and truthful in the range $v_i \geq \sum_j b_j/(1 - \alpha_i)$ for all $i$.

Proof: Pareto-optimality is trivial from proposition 2.4 since we charge bidders their full budget. We now prove truthfulness in the specified range. Since the values $v_i$ do not affect the payment or the allocation, it suffices to show that no manipulation of $b_i$ is profitable. Since we charge each bidder his total declared budget, it is clear that declaring $b'_i > b_i$ will lead to the bidder exceeding his budget. Thus it suffices to prove that no smaller declaration $b'_i < b_i$ is profitable. Let $u(z)$ be the utility obtained by bidder $i$ if he declares a budget of $b'_i = z$. Thus $u(z) = v_i \cdot z/(z + \sum_{j \neq i} b_j) - z$. It suffices to show that $u$ is monotonically increasing with $z$. To verify this, take the derivative with respect to $z$: $u'(z) = v_i \sum_{j \neq i} b_i/(z + \sum_{j \neq i} b_j)^2 - 1$. This derivative is non-negative, $u'(z) \geq 0$, if $v_i \geq (\sum_j b_j)^2/\sum_{j \neq i} b_j = \sum_j b_j/(1 - \alpha_i)$, as is indeed specified. □

3 The Clinching Auction for Players with Budgets

We formally describe the clinching auction for players with budgets, and show that it satisfies Pareto optimality, individual rationality, and truthfulness, when budgets are publicly known. The formal auction we describe is a direct mechanism whose outcome is chosen to be the outcome of Ausubel’s clinching auction, when budget-constrained players bid sincerely in it. In a high-level, Ausubel’s auction gradually increases a price parameter, and bidders keep decreasing their demands for items at this price. Whenever the combined demand of the other bidders decreases strictly below available supply, bidder $i$ “clinches” the remaining quantity at the current price. Thus different amounts of units are acquired by bidders at different prices, and the total payment of a bidder is the sum of the prices of all units that he clinched throughout the auction.

Before we begin the formal discussion, it might be useful to point out a subtle but important difference between the course of the clinching auction in the quasi-linear setting versus the budget setting: In the quasi-linear setting the demand curves of the bidders remain static, unchanged, throughout the course of the auction (the supply of-course changes). In the budget setting, demands themselves change, as previous clinching affect remaining budget, that in turn affects future demand. So demand as well as supply changes. This is also the reason why budgets must be public knowledge if players are strategic. To emphasize this effect that the change in setup has on the clinching auction, we add the word “adaptive” to its name.

Formally, the auction keeps for every player $i$ the current number of items $q_i$ already allocated to $i$, the current total price for these items $p_i$, and her remaining total budget $B_i = b_i - p_i$. The auction also keeps the global unit-price $p$ and the global remaining number of items $q$. The price $p$ gradually ascends as long as the total demand is strictly larger than the total supply, where the demand of player $i$ is defined by:

$$D_i(p) = \begin{cases} \left\lfloor \frac{B_i}{p} \right\rfloor & v_i > p \\ 0 & \text{otherwise} \end{cases}$$

If we were to keep the price ascending until total demand would be smaller or equal to the number items, and only then allocate all items according to the demands, then a player could sometimes gain by performing a “demand reduction”, thus harming truthfulness. Instead, following Ausubel’s method, we allocate items to player $i$ as soon as the total demand of the other players
decreases strictly below the number of currently available items, \( q \). In particular, if at some price \( p \) we have \( x = q - \sum_{j \neq i} D_j(p) > 0 \) then we allocate \( x \) items to player \( i \) for a unit price \( p \). At this point in the auction, the relevant variables are updated as follows: \( q_i \leftarrow q_i + x \), \( p_i \leftarrow p_i + p \cdot x \), \( b_i \leftarrow b_i - p \cdot x \), and \( q \leftarrow q - x \). This will ensure truthfulness. The global picture of such an auction is:

The Adaptive Clinching Auction (preliminary version):

1. While \( \sum_i D_i(p) > q \),
   (a) If there exists a player \( i \) such that \( D_{-i}(p) = \sum_{j \neq i} D_j(p) < q \) then allocate \( q - D_{-i}(p) \) items to player \( i \) for a unit price \( p \). Update all running variables, and repeat.
   (b) Otherwise increase the price \( p \), recompute the demands, and repeat.

2. Otherwise (hopefully \( \sum_i D_i(p) = q \)): allocate to each player her demand, at a unit-price \( p \), and terminate.

Note that step 1a does not change the amount of over demand, since both the total demand and the total supply are reduced by the same quantity (the number of items that player \( i \) gets). Therefore the only factor that affects the over demand is the price: as the price ascends the total over demand decreases. Thus, one would hope that when we reach step 2 we would indeed get \( \sum_i D_i(p) = q \), which will enable us to allocate all items at the end (a necessary condition for achieving Pareto optimality). However clearly this is not quite the case, because the demand functions are not continuous. The demand drops integrally, by definition, and may drop by several items at once. In particular, there are two potentially problematic change points: when the price reaches the value \( v_i \), and when the price reaches the remaining budget \( B_i \). The latter point is identified by using:

\[
D^+_i(p) = \lim_{x \to p^+} D_i(x),
\]
as, for \( p = B_i < v_i \), we have \( D_i(p) > 0 \) and \( D^+_i(p) = 0 \). Similarly, the former point is identified by using:

\[
D^-_i(p) = \lim_{x \to p^-} D_i(x),
\]
as, for \( p = v_i \leq B_i \), we have \( D^-_i(p) > 0 \) and \( D_i(p) = 0 \). We modify the above definition of the auction to use these more refined conditions: (1) the over demand is computed using \( D^+_i(p) \), since this ensures that we do not terminate with a price that is just a bit higher than the remaining budget of a player to whom we wish to allocate one last item, and (2) just before termination, if we are left with some non-allocated items, then this must have happened because the final price reached the value of some players (for such a player \( i \) we have \( D^-_i(p) > 0 \) and \( D_i(p) = 0 \)), which caused an abrupt decrease in her demand. These players are indifferent between receiving or not receiving an item, and so we can allocate to them all remaining items.

The Adaptive Clinching Auction (complete version):

1. While \( \sum_i D^+_i(p) > q \),
(a) If there exists a player $i$ such that $D^+_{-i}(p) = \sum_{j \neq i} D^+_{j}(p) < q$ then allocate $q - D^+_{-i}(p)$ items to player $i$ for a unit price $p$. Update all running variables (including the allocated and available quantities, the remaining budgets, and the current demands), and repeat.

(b) Otherwise increase the price $p$, recompute the demands, and repeat.

2. Otherwise ($\sum_i D^-_{i}(p) \geq q \geq \sum_i D^+_{i}(p)$):

(a) For every player $i$ with $D^+_{i}(p) > 0$, allocate $D^+_{i}(p)$ units to player $i$ for a unit-price $p$ and update all running variables.

(b) While $q > 0$ and there exists a player $i$ with $D_{i}(p) > 0$, allocate $D_{i}(p)$ units to player $i$, for a unit-price $p$, and update the running variables.

(c) While $q > 0$ and there exists a player $i$ with $D^-_{i}(p) > 0$, allocate $D^-_{i}(p)$ units to player $i$, for a unit-price $p$.

(d) Terminate.

Let us consider a short example to illustrate the above process. Suppose three items and three players with $v_1 = \infty, b_1 = 1, v_2 = \infty, b_2 = 1.9, v_3 = 1, b_3 = 1$. When the price is below 0.5, each player demands at least two items, and so, for every player, the other players demand more than three items. Therefore no allocations will take place, and the price will keep ascending. At $p = 0.5$, $D^+_{1}(0.5) = D^+_{3}(0.5) = 1$ (note that $D_1(0.5)$ and $D_3(0.5)$ are still 2). Thus, player 2 “clinches” one item for a price 0.5. Immediately after that, the demand of player 2 is updated to be 2. The available number of items is 2, and so no player can get any items. At a price 0.7 the demand of player 2 reduces to 1, but this still does not enable the auction to allocate any item to any player. The price keeps ascending until $p = 1$. At this point, $D^+_{1}(1) = 0, D^+_{2}(1) = 1, D^+_{3}(1) = 0$, and so the total demand reduces to be strictly below the number of available items (which is still 2). Thus we enter step 2. In 2a player 2 gets one item and in 2b player 1 gets one item. Note that we do not allocate any item to player 3, though $D^-_{3}(1) = 1$. Indeed, moving an item from 2 to 3, for example, will violate the Pareto optimality.

The following basic property of the auction almost immediately imply individual-rationality and truthfulness:

**Claim 3.1** The marginal utility of an item that is clinched at price $p$ is non-negative if and only if $p \leq v_i$.

**Proof:** If $p > v_i$ then by definition, since a player pays $p$ for the clinched item, its marginal utility is negative. Now assume that $p \leq v_i$. Whenever player $i$ gets $x$ items at a unit-price $p$ in steps 1a, 2a, and 2b in the auction, it follows that $x \leq D_{i}(p)$, where the demand is computed with respect to the remaining available budget. The definition of the demand function then implies that $B_{i} \geq x \cdot p$, hence the marginal utility is non-negative. If player $i$ gets an item in step 2c then $D_{i}(p) = 0$ and $D^-_{i}(p) > 0$. The structure of the demand function implies that this can happen only if $p = v_i$, and in addition the available budget at price $p$ is at least $D^-_{i}(p)$ times $p$. Thus in this case the player’s additional utility from those items is exactly zero. □

**Corollary 3.2** The adaptive clinching auction satisfies Individual Rationality, i.e. every truthful player obtains a non-negative utility.
Thus her items in step 2a or 2b (but not in 2c, since not all players in 2b were awarded their demand).

\[ \square \]

\textbf{Demand} \[ \square \]

Pareto improvement.

\[ \square \]

Step 2 then this implies that the remaining budget of player \( i \) means that the remaining available budget of \( j \) \((we will check the “no trade” condition of Prop. 2.5. We already showed property (a) Proof: \]

Claim 3.5 The adaptive clinching auction satisfies Pareto-optimality.

\[ \square \]

\textbf{Proof:} Define \( D(p) = \sum_i D_i(p) \) and define \( D^+(p) \) and \( D^-(p) \) similarly. Observe that these three functions are monotone non-increasing, and that \( D^-(p) = D(p) = D^+(p) \) for any continuity point of \( D(p) \). Moreover, if \( p^* \) is a discontinuity point of \( D(p) \) and \( D^+(p) > q \) for any \( p < p^* \) then \( D^-(p^*) \geq q \).

Suppose that the auction enters step 2 at a price \( p^* \). We wish to argue that \( D^-(p^*) \geq q \). Indeed, for any \( p < p^* \), at the beginning of step 1 we had \( D^+(p) > q \), and after step 1a this inequality is maintained (since if we allocate \( \Delta \) units to player \( i \) then the total demand and the number of available items both drop by \( \Delta \)). Therefore after step 1b we have \( D^+(p) \geq q \) (if \( p \) is a continuity point) or \( D^+(p) < q \) and \( D^-(p) \geq q \) (if \( p \) is a discontinuity point). In any case, if the auction enters step 2 then \( D^-(p^*) \geq q \), and the claim follows.\[ \square \]

\textbf{Claim 3.4} The adaptive clinching auction always allocates all items.

\[ \square \]

\textbf{Proof:} We will check the “no trade” condition of Prop. 2.5. We already showed property (a) \((\sum_i x_i = m)\) and it remains to show property (b). Fix any two players \( i \) and \( j \). We need to verify that, if \( j \) received at least one item, then \( i \)'s remaining budget at the end of the auction is smaller than \( j \)'s value. Consider the last price \( p \) at which player \( j \) received an item.

First suppose that \( p \) is not the price that ended the auction. In this case (step 1a), since \( j \) received an item, the auction rules imply that \( D^+_i(p) \) exactly equals the number of items left after player \( j \) was allocated her items. Since the auction allocates all items, and since it is IR, we get that each player \( i \neq j \) received after price \( p \) exactly \( D_i(p) \), her demand at \( p \). In particular, this means that the remaining available budget of \( i \) is at most \( p \) (otherwise the demand of \( i \) at \( p \) was higher – she could have bought one more item at a price lower than her value). On the other hand, \( v_j > p \), since \( j \) demanded items at \( p \), and we are done.

Now suppose that \( p \) is the price at which the auction ended. The auction rules imply that if \( i \) had \( D^+_i(p) > 0 \) then she received all this demand, and so by the same argument as above she does not have any remaining budget to buy an item from \( j \). A second case is \( D^+_i(p) = 0 \) and \( D_i(p) > 0 \). This implies that the remaining budget of player \( i \) at this step is \( B_i = p \). If player \( i \) received her demand \( D_i(p) \) then the argument of above still holds. If not, it must be that player \( j \) received her items in step 2a or 2b (but not in 2c, since not all players in 2b were awarded their demand).

Thus \( D_j(p) > 0 \) hence \( v_j > p = B_i \) and a Pareto improvement cannot take place. The last case is \( D_i(p) = 0 \) and \( D^-_i(p) > 0 \). Hence \( p = v_i \), and since \( v_j \geq p \) this again rules out the possibility of a Pareto improvement. \[ \square \]
4 Uniqueness of the Clinching Auction

In this section we show that the ascending clinching mechanism is essentially the only mechanism that is truthful, individually rational, and Pareto optimal for the setting of publicly known budgets. In the next section we utilize this result to show that there is no mechanism if the budgets are private.

Strictly speaking, we do not prove uniqueness for all possible budgets $b_1$ and $b_2$, but for “almost” all budgets. This is in a sense the best we can hope for, as, for example, for one item and $b_1 = b_2$ there are indeed multiple possible auctions (which are identical up to tie breaking). The following technical definition attempts to deal with this issue.

Let $S = (S_1, S_2)$ be a partition of $\{1, \ldots, m\}$. Given $b_1, b_2 \geq 0$, define $b^{k,S}_i$ recursively, for each $1 \leq k \leq m$: for $k = m$, $b^{m,S}_1 = b_1, b^{m,S}_2 = b_2$. For each $1 \leq k \leq m-1$, if $k \in S_1$ then: $b^{k,S}_1 = b^{k+1,S}_1, b^{k,S}_2 = b^{k+1,S}_2 - \frac{b^{k+1,S}_1}{k+1}$. If $k \in S_2$ then: $b^{k,S}_1 = b^{k+1,S}_1 - \frac{b^{k+1,S}_2}{k+1}, b^{k+1,S}_2 = b^{k+1,S}_2$. We say that $b_1$ and $b_2$ are $S$-generic if for each $1 \leq k \leq m$ we have that $b^{k,S}_1 \neq b^{k,S}_2$. We say that $b_1$ and $b_2$ are generic if they are $S$-generic for all $S$.

Notice that given any $b_1$ and $b_2$, a small perturbation will make them generic.

**Theorem 4.1** Let $A$ be a deterministic truthful mechanism for $m$ items and $2$ players with known budgets $b_1$ and $b_2$ that are generic. Assume that $A$ satisfies Pareto-optimality, individual-rationality, and no-positive-transfers. Then if $v_1 \neq v_2$ the outcome of $A$ coincides with that of the clinching auction.

The proof shows that all mechanisms that satisfy the requirements of the claim have the same outcome. Since the adaptive clinching auction satisfies all requirements of the claim, all other mechanisms coincide with it. We start with a useful lemma:

**Lemma 4.2** If $v_j < v_i$ and $v_j \leq \frac{b_i}{m}$ then player $i$ receives all items and pays $p_i = m \cdot v_j$ in any deterministic truthful mechanism that satisfies PO, IR, and NPT. In this case $j$’s payment, $p_j$, is exactly zero.

**Proof:** First consider the case $v_j < v_i < \frac{b_i}{m}$. In this case if player $i$ receives $x < m$ items then since by IR he pays at most $x \cdot v_i < \frac{m-1}{m} b_1$ he has left enough money to buy an item from player $j$ and pay him $v_j + \epsilon < v_i$, which contradicts PO. Thus player $i$ receives all items. Standard monotonicity arguments (see e.g. footnote 10 below) now imply that $i$ receives all items for any $v_i \geq \frac{b_i}{m}$ (when $v_j < \frac{b_j}{m}$).

If $v_j = \frac{b_j}{m}$ then for $v_i < \frac{m-1}{m-2} \cdot \frac{b_i}{m}$ it must be that player $i$ receives $x \geq m - 1$ items, otherwise if $x \leq m - 2$ then by IR $p_i \leq x \cdot v_i \leq \frac{(m-1)b_i}{m}$ and $b_i - p_i \geq \frac{b_i}{m} = v_j$, and by lemma 2.5 this contradicts PO since $v_i > v_j$. If $x = m - 1$ then by monotonicity player $i$ receives $m - 1$ items for any value in the interval $(\frac{b_i}{m}, v_i]$, therefore by truthfulness her payment $p_i$ is at most $\frac{(m-1)b_i}{m}$. But then again this contradicts PO as above. Thus player $i$ receives all items in this case as well.

To prove that the payments are as claimed first suppose that $v_j = 0$. By IR $p_j \leq 0$. For any declaration $v'_i > 0$ player $i$ receives all items (as argued above) and pays at most $p'_i \leq m \cdot v'_i$. Thus by truthfulness if $v_j = 0$ then $p_i \leq 0$. No-positive-payments requires $p_i + p_j \geq 0$ which implies $p_i = p_j = 0$ for the case $v_i > v_j = 0$.
For a general value $v_j$, since $j$ receives no items here as well, then truthfulness implies $p_j = 0$. Using the standard argument of the second-price auction we finally get that $p_i = m \cdot v_j$, and the claim follows. 

We continue with the main proof. Without loss of generality we assume throughout that $b_1 < b_2$. The proof is by induction on the number of items $m$, and we start with the base case $m = 1$.

**Lemma 4.3** All mechanisms for one item that satisfy the conditions of Theorem 4.1 have the same outcome if $v_1 \neq v_2$.

**Proof:** We show that the only possible mechanism is the following: the winner is the player $i$ that maximizes $\min(b_i, v_i)$. The winner pays the mechanism $\min(b_j, v_j)$, where $j$ is the other player, and the loser’s payment is exactly zero.\(^9\)

It is easy to verify that the above mechanism satisfies the required properties. We now prove that this is the only possible mechanism. If $\min(v_1, v_2) \leq b_1$ then the claim follows from lemma 4.2. Otherwise assume $v_1, v_2 > b_1$.

We show that player 2 must win the item. First observe that if $v_1 < \min(v_2, b_2)$ then the only Pareto-optimal outcome allocates the item to 2 (in the other allocation player 2 can buy the item from 1, and they are both better off). Suppose that there exists some value $v'_1 > b_1$ such that 1 wins the item even though $v_2 > b_1$. By feasibility 1’s payment in this case is at most $b_1$, and 1 has positive utility from declaring $v'_1$. Thus when 1’s true value is $b_1 < v_1 < \min(v_2, b_2)$ he can declare $v'_1$ and improve his utility, contradicting truthfulness.

Therefore for any $v_2 > b_1$ player 2 must be the winner. Player 1’s payment must be exactly zero by truthfulness since his payment must be equal to the case when he declares $v'_1 < b_1$. This also implies that player 2’s payment is the minimal possible value he needs to declare in order to win, i.e. $\min(b_1, v_1)$, and the claim follows. \(\Box\)

We now continue the induction, assuming uniqueness for $m - 1$ items, and proving uniqueness for $m$ items. The logic is as follows. We start with some mechanism $A$ for $m$ items that satisfies the conditions of Theorem 4.1. We then explicitly describe the allocation and payments of $A$ on all instances, except for instances of the form $v_1, v_2 \geq \frac{b_1}{m}$. To characterize $A$’s behavior in this domain, we use $A$ to construct a new mechanism $A_{m-1}$ for $m - 1$ items and different budgets. At the beginning $A_{m-1}$ will only be defined on $v_1, v_2 \geq \frac{b_1}{m}$. We will show that the outcome of $A$ on instances where $v_1, v_2 \geq \frac{b_1}{m}$ is defined by the outcome of $A_{m-1}$.

Now we would like to finish the proof by claiming that $A_{m-1}$ is unique, by the induction hypothesis. However, since $A_{m-1}$ is not defined on all the domain of possible valuations, we cannot directly apply the induction hypothesis, as there might be other mechanisms if the domain of possible valuations is restricted. To overcome this, we will extend $A_{m-1}$, and define it on all valuations in the domain. Then we will show that $A_{m-1}$ satisfies all conditions of Theorem 4.1, hence it is unique by the induction hypothesis. Now we can uniquely determine the outcome of $A$ on all possible valuations, and in particular in the domain $v_1, v_2 \geq \frac{b_1}{m}$, as needed.

Let us now define the mechanism $A_{m-1}$. $A_{m-1}$ works on budgets $b'_1 = b_1$ and $b'_2 = b_2 - \frac{b_1}{m}$. Notice that $b'_1$ and $b'_2$ are generic, and that now it is not necessarily true that $b'_1 \leq b'_2$. We start

\(^9\)Notice that if $b_1$ and $b_2$ are not generic, i.e., $b_1 = b_2$, then indeed this auction is not uniquely defined as if $v_1, v_2 > b_1 = b_2$ we can break ties in favor of both players, resulting in multiple possible outcomes. Also notice that this mechanism is indeed identical to the clinching auction.
by defining $A_{m-1}$ on instances where $v_1, v_2 > \frac{b_1}{m}$; denote the outcome of $A$ for $v_1$ and $v_2$ by $(\bar{x}, \bar{p})$, where $x_i$ is the amount that $i$ gets, and $p_i$ is his payment. Let the outcome of $A_{m-1}$ be $(x_1, p_1)$ for player 1 (i.e., as in $A$), and for player 2 let the outcome be $(x_2, p_2 - \frac{b_1}{m})$. In particular, observe that given the outcome of $A_{m-1}$ on valuations in this domain, we can deduce the outcome of $A$ on the same valuations.

We now extend the definition of $A_{m-1}$ for valuations where $\min(v_1, v_2) \leq \frac{b_1}{m}$. In this case we allocate all items to the bidder with the highest value, and his payment is $m-1$ times the value of the other player.

**Lemma 4.4** $A_{m-1}$ yields a valid outcome, and is Pareto optimal and truthful.

Before proving this lemma we show the following helpful lemmas:

**Lemma 4.5** Let $A$ be a mechanism for $m$ items that is Pareto optimal, individually rational, and truthful. Suppose that $\min(v_1, v_2) > \frac{b_1}{m}$. Then, a player that wins $x$ items pays at least $x \cdot \frac{b_1}{m}$.

**Proof:** Suppose by contradiction that there exist $(v_1, v_2)$ in which some player $i$ gets $x \geq 1$ items and pays $t < x \cdot \frac{b_1}{m}$. Consider now a different valuation $v_i'$ such that $t/x < v_i' < \frac{b_1}{m}$. By Lemma 4.2 $i$ is allocated no items when he declares are $v_i'$ and the other player declares the same as before. Here $i$ will be better off by declaring $v_i$ instead of $v_i'$, since he will be allocated $x$ items and will get a positive utility: $x \cdot v_i' - t > 0$, contradicting truthfulness.

**Lemma 4.6** Let $A$ be a mechanism for $m$ items that is Pareto optimal, individually rational, and truthful. Suppose that $v_2 > \frac{b_1}{m}$. Then, player 2 wins at least one item.

**Proof:** Suppose that there is a declaration $v_1$ such that, when the players declare $(v_1, v_2)$, player 1 win all items. By Lemma 4.5 the payment of player 1 is at least $m \cdot \frac{b_1}{m} = b_1$. His payment is exactly $b_1$ since this is his budget. By truthfulness, in any declaration $v_1' > \frac{b_1}{m}$ he must still win all items (player 2 still declares $v_2$). Fix $v_1'$ such that $\min(v_2, b_2) > v_1' > \frac{b_1}{m}$. From above we get that player 1 gets all items when the declarations are $(v_1', v_2)$. However this contradicts pareto-optimality, using claim 2.5, since $v_2 > v_1'$ but $p_2 = 0 < b_2 - v_1'$.

**Lemma 4.7** Let $A$ be a mechanism for $m$ items that is Pareto optimal, individually rational, and truthful. Suppose that $v_1 > \frac{b_1}{m}$. Then, if player 2 wins exactly one item he pays exactly $\frac{b_1}{m}$.

**Proof:** Fix some $v_2$ such that, when the declaration is $(v_1, v_2)$, player 2 gets $x_2 = 1$ and pays some $p_2$. By claim 4.5, $p_2 \geq \frac{b_1}{m}$. Now fix some $v_2'$ such that $v_2 > v_2' > \frac{b_1}{m}$. Suppose that in the declaration $(v_1, v_2')$ player 2 gets $x_2'$ and pays $p_2'$. It is well-known\(^{10}\) that truthfulness implies that $x_2' \leq x_2$. By claim 4.6 $x_2' \geq 1$, and therefore we must have $x_2' = 1$. Truthfulness now implies that $p_2 = p_2'$. Therefore we have $v_2' \geq p_2 \geq \frac{b_1}{m}$. Since this is true for any $v_2' > \frac{b_1}{m}$ we get that $p_2 = \frac{b_1}{m}$, as claimed.

**Proof:** (of Lemma 4.4) During the proof we abuse notation a bit and identify the outcome of $A$ with $A'$, and the outcome of $A_{m-1}$ with $A_{m-1}$. We break the proof into several claims.\(^{10}\)

\(^{10}\) A short proof, based on the W-MON condition of Bikhchandani et al. (2006), is: from truthfulness we have $v_2 \cdot x_2 - p_2 \geq v_2 \cdot x_2' - p_2'$ since when the true type is $v_2$ the player will not benefit from declaring $v_2'$. Similarly, $v_2' \cdot x_2' - p_2' \geq v_2 \cdot x_2 - p_2$. Combining, we get $v_2(x_2' - x_2) \geq p_2 - p_2' \geq v_2(x_2' - x_2)$, and since $v_2 < v_2'$ it follows that $x_2' \leq x_2$. 

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Claim 4.8 $A_{m-1}$ yields a valid outcome.

Proof: We show that $A_{m-1}$ is individually rational, i.e., a player that receives no items pays no items. The other properties are straight-forward. If $\min(v_1, b_1) \leq \frac{b_i}{m}$, then we conduct a second price auction, and the loser pays nothing. Else, if player 1 is allocated no items in $A_{m-1}$, then he pays nothing, since $A$ is individually rational and 1 gets nothing also in $A$. Consider the case where player 2 is allocated no items in $A_{m-1}$. It means that it was allocated exactly one item in $A$, and by Lemma 4.7 his payment is $\frac{b_i}{m}$ in $A$, hence in $A_{m-1}$ his payment is 0.

Claim 4.9 $A_{m-1}$ is Pareto optimal.

Proof: Consider first the case where $v_1, v_2 > \frac{b_i}{m}$. By claim 2.5, it is enough to show two things: (1) If $v_1 > v_2$ then $p'_1 > b'_1 - v_2$: since $A$ is Pareto-optimal then $p_1 > b_1 - v_2$, and since $p'_1 = p_1$ and $b'_1 = b_1$ the claim follows; and (2) If $v_2 > v_1$ then $p'_2 > b'_2 - v_1$, or, equivalently, $v_1 > b'_2 - p'_2$: since $A$ is Pareto-optimal then $v_1 > b_2 - p_2$, and since $b'_1 - p'_2 = b_2 - p_2$ the claim follows.

Now consider the case where $\min(v_1, v_2) \leq \frac{b_i}{m}$. Let $b'_i = \min(b'_1, b'_2)$. First, observe that we have that if $b'_i = b'_1$ then $\frac{b_i}{m} \leq \frac{b_i}{m-1}$, since $b'_i = b'_1$. For $b'_i = b'_2 = b_2 - \frac{b_i}{m}$, we also have that $\frac{b_i}{m-1} = \frac{b_i - \frac{b_i}{m}}{m-1} \geq \frac{b_i - \frac{b_i}{m}}{m-1} > \frac{b_i}{m}$. Hence in this range, by Lemma 4.2, it is Pareto optimal to allocate all items to the bidder with the highest value, as $A_{m-1}$ indeed does.

Claim 4.10 $A_{m-1}$ is truthful.

Proof: Once again we consider the several different cases. Start with the case where $v_1, v_2 > \frac{b_i}{m}$, and suppose player $i$ declares $v'_i > \frac{b_i}{m}$ instead (and is allocated $x'_i$ items and pays $p'_i$). Clearly, $i \neq 1$, as the allocation and payment of player 1 are the same as in $A$, and $A$ is truthful. Suppose $i = 2$ is better off declaring $v'_2$: $v_2(x_2) - p_2 < v_2(x'_2) - p'_2$. Observe that in $A$ we have that: $v_2(x_2 + 1) - (p_2 + \frac{b_i}{m}) < v_2(x'_2 + 1) - (p'_2 + \frac{b_i}{m})$, a contradiction to the truthfulness of $A$.

Suppose that $v_1, v_2 > \frac{b_i}{m}$, and that player $i$ declares $v'_i < \frac{b_i}{m}$ instead. Notice that $x'_i = 0$, so $i$ cannot increase his profit from declaring $v'_i$.

In the case where $\min(v_1, v_2) \leq \frac{b_i}{m}$ player $i$ is not better off declaring $v'_i < \frac{b_i}{m}$, as in this range we are essentially conducting a second price auction, which is truthful.

Finally, suppose $\min(v_1, v_2) \leq \frac{b_i}{m}$. Consider player $i$ that declares $v'_i > \frac{b_i}{m}$. Suppose $v_j > \frac{b_i}{m}$, where $j$ is the other player. Observe that if $i$ wins some items, then by Lemma 4.5 $j$ has to pay at least $\frac{b_i}{m}$ for every item he wins, which is more than is value. If $v_j < \frac{b_i}{m}$, then we conduct a second price auction, regardless of what $i$ declares, and this auction is truthful. By the induction hypothesis, we have that $A_{m-1}$ is unique. By our discussion, this is enough to prove the uniqueness of $A$ and this concludes the proof of the theorem.

5 An Impossibility Result for Private Budgets

Once the public-budgets case is completely analyzed, the impossibility for private budgets follows quite easily. We start with the case of two players and then show the general case.

Theorem 5.1 There is no deterministic truthful auction that satisfies Pareto-optimality, individual-rationality, and no-positive-transfers, for two players with private budgets.
Proof: An auction $A$ for private budgets is also truthful if budgets are public. By our uniqueness result for two players with public budgets, we therefore conclude that the outcome of $A$ must be the same as the outcome of the clinching auction.

Consider two instances for the clinching auction. First, $b_1 = 1, v_1 = \infty, b_2 = 1 + \sum_{k=2}^{m-2} \frac{1}{k} - \delta, v_2 = \infty$, for some small $\delta > 0$. ($\delta$ is chosen to make $b_1$ and $b_2$ generic). For each of the first $m-1$ items, the clinching auction will allocate the item to player 2 and will charge $\frac{1}{k}$ for the $k$’th item. Then, at the $k$’th item, player 1’s budget is finally larger than player 2’s free budget, so player 1 wins the last item with a payment of $1 - \delta$.

Second, $b'_1 = 1 + \epsilon$, for small enough $\epsilon$, and the other parameters are as above. The resulting allocation is the same as above, but player 2 is charged $\frac{1+\epsilon}{k}$ for the $k$’th item (for $k > 1$). Thus, when the auction allocates the last item, player 2’s free budget is smaller than before: $1 - \delta - \sum \epsilon_k$. This is also the payment of player 1.

Therefore player 1 is allocated one item in both cases, but his payment is smaller in the second case, so his utility is larger. Now, as argued in the first paragraph of this proof, $A$’s outcome is the same as the outcome of the clinching auction for both cases. Therefore when the players’ types are as in the first case, player 1 can improve his resulting utility from the mechanism $A$ by declaring a false budget $b'_1 = 1 + \epsilon$. This false budget will change the outcome of $A$ to be that of the clinching auction for the second case, and will thus increase player 1’s utility, which contradicts truthfulness. □

The contradiction in the proof was obtained by reporting a budget which is higher than the true budget. The follow-up paper Bhattacharya et al. (2010) shows that there are cases where it is profitable to declare a budget lower than the true budget (though for a divisible item only higher budgets can be profitable deviations).

Corollary 5.2 There is no deterministic truthful auction that satisfies Pareto-optimality, individual-rationality, and no-positive-transfers, for any number of players with private budgets.

Proof: Suppose by contradiction that there exists an auction $A_n$ for $n > 2$ players with private budgets that satisfies all properties of the claim. Then there is an auction $A_2$ for two players with private budgets that satisfies all properties of the claim: upon receiving the declarations of the two players, $A_2$ adds $n-2$ players that have a budget of zero and a value of zero, and determines the allocation and payments of the two “real” players to be the same as their allocation and payments in $A_n$ with the $n-2$ dummy players. Since $A_n$ satisfies all properties of the claim then $A_2$ satisfies all properties of the claim as well, contradicting Theorem 5.1. □

6 Revenue Considerations

Up to now we have discussed the efficiency properties of the clinching auction for players with budgets. We now examine its revenue properties. We will compare the revenue of the clinching auction to the revenue of a non-discriminatory monopoly, that knows the budgets and values of the players, and has to determine a single unit-price at which items will be sold. To strengthen our result and simplify the analysis at the same time, we allow the monopoly (but not the mechanism!) to sell also fractions of the good, and not just integer quantities.

The approach of comparing an auction’s revenue to the optimal fixed-price revenue was initiated by Goldberg, Hartline, Karlin, Saks and Wright (2006). In the context of auctions with budget limitations it was used by Borgs et al. (2005) and Abrams (2006). In particular, Abrams (2006)
showed that the optimal monopoly revenue is always at least half of the optimal multi-price revenue, that may charge different prices from different players.\footnote{The argument is based on the following claim: if in the competitive equilibrium there is more than a single winner, then the revenue of this outcome is at least half of the optimal revenue (the maximal payment that satisfies individual rationality: \( p_i \leq b_i \) and \( p_i \leq x_i \cdot v_i \)). Let us sketch the proof of this. Let \( p \) be the equilibrium price. Split the bidders to those with \( v_i > p \) and those with \( v_i \leq p \). The equilibrium revenue is \( m \cdot p \). All bidders in the first set pay their full budget anyway in the equilibrium. We can never get more than a total of payment \( m \cdot p \) from all bidders in the second set (since \( v_i \leq p \)). Thus the optimal revenue is at most \( 2m \cdot p \).} Thus, comparing the revenue of the auction to any other revenue criteria can yield a ratio which may be smaller by a constant factor of at most 1/2.

To formally define our benchmark for revenue, let a fractional allocation be a real vector \( x = (x_1, \ldots, x_n) \), where for each \( i \), \( x_i \geq 0 \), and \( \Sigma_i x_i \leq m \). Given a fractional assignment \( x \), define the monopoly revenue from \( x \) to be \( \Sigma_i x_i \cdot p^*(x) \), where \( p^*(x) \) is the largest price that satisfies, for each \( i \) with \( x_i > 0 \), \( v_i \geq p^*(x) \), and \( b_i \geq x_i \cdot p^*(x) \). Define the optimal monopoly revenue to be the supremum over all fractional assignments \( x \) of the monopoly revenue from \( x \). Let \( x^* \) be the fractional allocation that obtains this optimal monopoly revenue, and \( p^* = p^*(x^*) \). Our analysis uses the following “bidder dominance” parameter:

\[
\beta = \max_{i=1,\ldots,n} \frac{x_i^*}{\sum_{j=1}^{n} x_j^*}.
\]

If \( \beta = 1 \) then all items are sold to one single player. In this case, one bidder stands out, and the monopoly prefers to focus on him and extract all his surplus by setting a high price. Thus it is intuitively clear that the clinching auction cannot hope to extract a large fraction of the monopoly’s revenue since there is no real competition. As \( \beta \) decreases, this “best” bidder faces more competition, and the clinching auction raises a larger fraction of the monopoly’s revenue. Formally, we show:

**Theorem 6.1** The revenue of the clinching auction is at least a fraction of \( \frac{m}{m+n} \cdot (1 - \beta) \) of a fixed-price monopoly’s optimal revenue.

Note that this theorem gives interesting bounds only when the number of items \( m \) is much larger than the number of bidders \( n \) (i.e. we approach the case of a divisible item). This is a consequence of choosing to compare to a monopoly that can decide on fractional allocations of items to buyers. For example, suppose there is one item \((m=1)\) and every bidder \( i = 1, \ldots, n \) has \( v_i = n \) and \( b_i = 1 \). The monopoly will choose \( p^* = 1 \) and \( x_i^* = 1/n \) for every bidder \( i = 1, \ldots, n \), yielding a revenue of \( n \), while the clinching auction has revenue 1 because it has to give the item integrally to one of the players. This shows that the bound in the theorem cannot be significantly improved.

Alternatively, we can compare to a monopoly that is also restricted to assign the items integrally. In this case, an alternative statement is that the revenue of the clinching auction is at least a fraction of \( \frac{m}{2(m+n)} \cdot (1 - \beta) \) of the monopoly’s revenue. This claim follows using virtually the same proof we describe below (instead of Claim 6.2 we need to argue that without loss of generality the monopoly allocates at least half of the items).

We note that it is necessary to have some integrality factor even when comparing to a monopoly that assigns the items integrally. To demonstrate this, consider the following example. Suppose the number of items and bidders is equal, and all bidders have a budget 1 and value \( m \). The monopoly prefers to focus on the most valuable bidder and extract all his surplus by setting a high price. Thus the optimal revenue is at most 2 times the fixed-price monopoly’s optimal revenue.
player, for a price of 1/2, since at this price $D_i^+(1/2) = 1$ for every player $i$. Thus, there is a ratio of 1/2 between the revenue of the clinching auction and the monopoly’s revenue.

**Proof:** (of Theorem 6.1)

We denote the optimal monopoly price by $p^*$, and the fractional assignment that maximizes the optimal monopolist price by $x^* = (x_1^*, \ldots, x_n^*)$. We show two claims:

**Claim 6.2** It can be assumed without loss of generality that all items are allocated in the fractional assignment that maximizes the optimal monopolist price. I.e., $\sum_i x_i^* = m$.

**Proof:** Assume that $\sum_i x_i^* < m$. Let $W = \{ i \mid v_i \geq p^* \}$ and $B = \sum_{i \in W} b_i$. Since the unit-price is $p^*$, any player $i$ with $v_i < p^*$ must have $x_i^* = 0$, hence the optimal monopoly price is at most $B$. Additionally, for any $i \in W$ we must have $x_i^* = b_i/p^*$ since otherwise we can increase the quantity that $i$ gets, contradicting the fact that $x^*$ maximizes the revenue. This implies that $\sum_{i \in W} b_i/p^* = \sum_{i \in W} x_i^* < m$, hence $p^* > B/m$. Now, by setting $p = B/m$ and $x_i = b_i/p$ for any $i \in W$ (note that $v_i \geq p^* > B/m = p$), we get revenue exactly $B$, and $\sum_i x_i = m$, thus the claim follows. \qed

**Claim 6.3** No player clinches an item before the price reaches $\tilde{p} = \frac{m}{m+n} \cdot (1 - \beta) \cdot p^*$.

**Proof:** We will show that, for each player $i$, $\sum_{j \neq i} D_j(\tilde{p}) \geq m$, which implies the claim. Let $W = \{ j \mid x_j^* > 0 \}$, and $W_i = W \setminus \{ i \}$. For any $j \in W$, $v_j \geq p^* > \tilde{p}$, hence $D_j(\tilde{p}) = \lfloor \frac{b_j}{\tilde{p}} \rfloor$. We therefore have

$$\sum_{j \neq i} D_j(\tilde{p}) \geq \sum_{j \in W_i} D_j(\tilde{p}) = \sum_{j \in W_i} \lfloor \frac{b_j}{\tilde{p}} \rfloor \geq \sum_{j \in W_i} \left( \frac{b_j}{\tilde{p}} - 1 \right) \geq \sum_{j \in W_i} \frac{b_j}{\tilde{p}} - n$$

We next note that $\sum_{j \in W_i} x_j^* = m - x_i^* \geq m - \beta \cdot m = m(1 - \beta)$. This gives us:

$$\sum_{j \neq i} D_j(\tilde{p}) \geq \sum_{j \in W_i} \frac{b_j}{\tilde{p}} - n = \frac{m + n}{m} \cdot \frac{1}{1 - \beta} \cdot \sum_{j \in W_i} \frac{b_j}{p^*} - n \geq \frac{m + n}{m} \cdot \frac{1}{1 - \beta} \cdot m(1 - \beta) - n = m$$

which proves the claim. \qed

We now prove Theorem 6.1. By claim 6.2 we may assume that the optimal revenue is achieved by allocating all items and thus the optimal monopoly revenue is at most $m \cdot p^*$. The adaptive clinching auction sells all items (by claim 3.4), and by claim 6.3 each item is sold for a price of at least $\tilde{p} = \frac{m}{m+n} \cdot (1 - \beta) \cdot p^*$. Thus the revenue of the adaptive clinching auction is at least $m \cdot \tilde{p} = \frac{m}{m+n} \cdot (1 - \beta) \cdot (m \cdot p^*)$. \qed

7 The Infinitely-Divisible Good Setting

While the adaptive clinching auction may be applied in the infinitely divisible setting by treating it as a continuous time process, the analysis is not straight-forward. In this section we rely on this
process to obtain an explicit closed-form auction for a divisible good setting, and we directly prove that it is truthful and Pareto-optimal. We limit ourselves to the case of two bidders. We then show that if the budgets are equal then this auction is unique among all anonymous auctions, and we use this to derive a general impossibility result for anonymous mechanisms in the private-budget case.

7.1 A mechanism for known budgets

We construct a truthful and Pareto-optimal mechanism for two bidders with publicly-known budgets. We start by analyzing two special cases, that will be used later on as building blocks for the general mechanism.

First special case: only one bidder with a budget limit. We first look at the case where only one of the players is budget-limited. Assume that $b_1 = 1$ (this is w.l.o.g) and $b_2 = \infty$. Let us overview the course of the adaptive clinching auction for this case. As long as the price $p$ is below 1 and below $\min(v_1, v_2)$, both players demand all the quantity, and so no clinching occurs. If $\min(v_1, v_2) \leq 1$ then the player $i$ with the minimal value will drop out when the price will reach her value, and the other player will get the entire quantity and will pay the lower value. Otherwise assume that $\min(v_1, v_2) > 1$. When the price exceeds 1, player 1 starts reducing her demand to quantities smaller than 1 (recall that $D_1(p) = b_i/p$). Therefore player 2 starts clinching the quantity that is not being demanded anymore by player 1. The total quantity clinched up to price $p$ is $1 - D_1(p) = 1 - 1/p$ and thus player 2 clinches $d(1 - D_1(p))/dp = 1/p^2$ units at marginal price $p$. The total payment of player 2 up to price $p$ is obtained by integrating the product. This continues until the price reaches $\min(v_1, v_2)$ (recall that player 2 has infinite budget, hence she never reduces her demand). Once we reach the point $p = \min(v_1, v_2)$, the lower player drops, and the larger player gets the remaining quantity at the current unit-price. This leads us to “guess” the following mechanism for this special case:

**Definition 7.1 (Mechanism A)**

- If $\min(v_1, v_2) \leq 1$ then the high player gets everything at the second price: $x_i = 1, p_i = v_j$ (and $x_j = 0, p_j = 0$), where $v_i > v_j$.
- Otherwise, if $v_2 \geq v_1$ then the high non-budget-limited player gets everything $x_2 = 1$ and pays $1 + \ln v_1$.
- Otherwise, if $v_1 > v_2$ then the high player gets $x_1 = 1/v_2$ and pays $p_1 = 1$, while the non-budget-limited player gets $x_2 = 1 - 1/v_2$ and pays $p_2 = \ln v_2$.

We give an explicit proof that Mechanism A indeed satisfies Pareto-optimality and truthfulness. In the proof, we use a slightly weaker assumption instead of $b_2 = \infty$, a relaxation that will become important in the sequel.

**Proposition 7.2** Fix any two budgets $b_1 \leq b_2$. Then, mechanism A is Pareto-optimal and individually-rational, and,

1. It is a dominant-strategy for player 1 to declare her true value.
2. If $v_2 \leq e^{b_2-1}$ then it is a dominant-strategy for player 1 to declare her true value. More precisely, let $u_2(z)$ denote player 2’s resulting utility when she declares $z$. Then $u_2(v_2) \geq u_2(z)$ for any real number $z$.

**Proof:** Pareto-optimality follows directly from proposition 2.4 since in the first two cases the low bidder gets allocated 0, and in the last case, the high bidder has his budget exhausted.

Let us start by looking at the incentives of bidder 1. If $v_2 \leq 1$ then he is faced with exactly two possibilities $x_1 = 1, p_1 = v_2$ and $x_1 = 0, p_1 = 0$. It is clear that he prefers the former if and only if $v_1 \geq v_2$, which is what happens with the truth. If $v_2 > 1$ then he is faced with two possibilities: either declare some $z \leq v_2$ in which case he gets $x_1 = 0, p_1 = 0$ or declare some $z > v_2$ and get allocated $x_1 = 1/v_2, p_1 = 1$. His utility in the first case is $u_i = 0$ and in the second $u_i = v_1/v_2 - 1$, which is positive iff $v_1 > v_2$ and given to him by the mechanism when telling the truth $z = v_1$.

Now for bidder 2. The case $v_1 \leq 1$ is as before. Otherwise he may declare either $z < v_1$ getting $x_2 = 1 - 1/z, p_2 = \ln z$ or declaring $z \geq v_1$ getting $x_2 = 1, p_2 = 1 + \ln v_1$. In the first case his utility is at most $u_2(z) = v_2 - v_2/z - \ln z$ (it is exactly this term if $p_2 \leq b_2$, otherwise it is smaller). This term for $u_2(z)$ is maximized for $z = v_2$ (by solving for $du_2/dz = 0$). Thus in the first case his utility is at most $v_2 - 1 - \ln v_2$. In the second case his utility at most $u_2 = v_2 - 1 - \ln v_1$. If $v_2 < v_1$ then the former term is larger than the latter term, and indeed by declaring $z = v_2$ the player obtains a utility exactly equal to $v_2 - 1 - \ln v_2$ since when $z = v_2$ we have $p_2 = \ln v_2 < \ln e^{b_2-1} < b_2$. If $v_2 \geq v_1$ then the latter term is better, and indeed by declaring $z = v_2$ the player obtains a utility exactly equal to $v_2 - 1 - \ln v_1$ since in this case $p_2 = 1 + \ln v_1 \leq 1 + \ln v_2 \leq 1 + \ln e^{b_2-1} = b_2$. Thus declaring $z = v_2$ obtains maximal utility, no matter what is $v_1$.

Individual-rationality follows from truthfulness, since a player can always obtain a zero utility by declaring $v_i = 0$. \qed

**Corollary 7.3** Mechanism A is Pareto-optimal and truthful, assuming only one bidder is budget-constrained.

**Second special case: bidders with equal budgets.** The second special case we analyze is when the budgets are equal. Assume without loss of generality that $b_1 = b_2 = 1$ and $v_1 \leq v_2$. In addition, it will be useful for the sequel to explicitly denote the initial quantity by $Q$ (and not to assume $Q = 1$).

We again “guess” the correct mechanism by looking at the course of the adaptive clinching auction. Similarly to before, while $p \leq 1/Q$ no clinching occurs since each player demands all available quantity. At this point, the demand of both players is equal to available quantity, and hence from this point on both players will start clinching. Calculating the exact rate at which the clinching occurs is slightly more involved in this case. Let $D_i(p), b_i(p)$ denote the current demand and remaining budget of player $i$ at price $p$, and let $q_i(p)$ denote the total quantity that player $i$ have received up to price $p$. When the price reaches $\min(v_1, v_2)$, the lower player drops and the higher player receives the remaining quantity, but before this point the two players are completely identical, so we can remove the subscript $i$ from the three functions. We have

$$D(p) = \frac{b(p)}{p}, \quad b'(p) = -q'(p) \cdot p$$

directly from the definition of the adaptive clinching auction. It will turn out useful to construct the three functions so that clinching will continuously occur, for all prices $p \geq 1/Q$. For this to happen,
we need that the current demand of each player will always be exactly equal to the current available quantity (since in such a case, and only in such a case, when a player decreases her demand, the other player performs clinching). This means:

\[ D(p) = Q - 2 \cdot q(p) \]

Solving these three equations, we get:

\[ q(p) = \frac{Q}{2} - \frac{1}{2 \cdot Q \cdot p^2}, \quad b(p) = \frac{1}{Q \cdot p} \]

We next show explicitly that using these functions will indeed yield Pareto optimality and truthfulness. Moreover, in the sequel (Theorem 7.9) we show that this is the unique anonymous mechanism that is Pareto-optimal and truthful.

**Definition 7.4 (Mechanism B)** Assume that \( b_1 = b_2 = 1 \) and \( v_1 \leq v_2 \). Assume also that the initial available quantity is \( Q \).

- If \( v_1 \leq 1/Q \) then the high player gets everything at the second price: \( x_2 = Q, p_2 = v_1 \cdot Q \) (and \( x_1 = 0, p_1 = 0 \)).

- Otherwise, the low player gets \( x_1 = Q/2 - 1/(2 \cdot Q \cdot v_1^2) \) and pays \( p_1 = 1 - 1/(Q \cdot v_1) \) and the high player gets \( x_2 = Q/2 + 1/(2 \cdot Q \cdot v_1^2) \) and pays \( p_2 = 1 \).

**Proposition 7.5** Mechanism B is Pareto-optimal, individually-rational, and truthful, in the case of publicly known and equal budgets.

**Proof:** Pareto-optimality follows directly from proposition 2.4: in the first case the high player gets all the quantity, and in the second case the budget of the high player is exhausted.

Let us consider the incentives of one bidder with value \( v_i \) when the other bids a fixed value \( v_j \). If \( v_j \leq 1/Q \) then bidder \( i \) can choose between declaring \( z \leq v_j \) in which case \( x_i = 0, p_i = 0 \) and thus \( u_i = 0 \) (in case of tie, if \( x_i = 1, p_i = v_j \) then we still have \( u_i = 0 \)) to bidding \( z > v_j \) in which case \( x_i = Q, p_i = v_j \cdot Q \) and thus \( u_i = (v_i - v_j)Q \). The latter is better if and only if \( v_i > v_j \), and by bidding \( z = v_j \) player \( i \) gets the better option.

If \( v_j > 1/Q \), then bidder \( i \) can choose between declaring \( z < v_j \) in which case \( x_i = Q/2 - 1/(2 \cdot Q \cdot z^2), p_i = 1 - 1/(Q \cdot z) \) to bidding \( z > v_j \) in which case \( x_i = Q/2 + 1/(2 \cdot Q \cdot v_j^2), p_i = 1 \). Thus the utility when bidding \( z < v_j \) is \( v_i(Q/2 - 1/(2 \cdot Q \cdot z^2)) - (1 - 1/(Q \cdot z)) \), and this is maximized by \( z = v_i \). Thus the utility when bidding \( z < v_j \) is at most \( v_i(Q/2 - 1/(2 \cdot Q \cdot v_j^2)) - (1 - 1/(Q \cdot v_j)) \) (call this \( u^{(L)}(i) \)), and the utility when bidding \( z > v_j \) is exactly \( v_i(Q/2 + 1/(2 \cdot Q \cdot v_j^2)) - (1 - 1/(Q \cdot v_j)) \) (call this \( u^{(H)}(i) \)).

A short calculation shows that \( u^{(L)} > u^{(H)} \) if and only if \( v_i < v_j \). Therefore: (1) if \( v_i < v_j \) then a player will maximize his utility by obtaining a utility equal to \( u^{(L)} \), which can be obtained by declaring \( z = v_i \), and (2) if \( v_i > v_j \) then a player will maximize his utility by obtaining a utility equal to \( u^{(H)} \), which can be obtained by declaring \( z = v_i \). Thus no matter what is \( v_j \), declaring \( v_i \) will maximize player \( i \)'s utility. This proves truthfulness.

Individual-rationality follows from truthfulness, since a player can always obtain a zero utility by declaring \( v_i = 0 \). \( \square \)
The general case: bidders with arbitrary budgets. We now reach the case of general budgets, and again wish to examine the course of the adaptive clinching auction before constructing the closed-form mechanism. Assume that \( b_1 = 1 < b_2 \). When the price just crosses the point \( p = 1 \) the situation is similar to the first special case from above: player 2 still demands all quantity so player 1 does not perform clinching, and player 1 starts reducing her demand, so player 2 starts to clinch. Using the equations found in the first special case from above, the total clinched quantity of player 2 at price \( p \) is \( q_2(p) = 1 - 1/p \), and her remaining budget is \( b_2(p) = b_2 - \ln p \). This situation continues until the point where the available quantity at price \( p \) equals the demand of player 2 at that price, which can be found by solving:

\[
\frac{b_2 - \ln p}{p} = \frac{b_2(p)}{p} = D_2(p) = 1 - q_2(p) = \frac{1}{p}
\]

and the solution is \( p^* = e^{b_2 - 1} \). To verify, note that at this price the available quantity is \( 1/p^* \), and the remaining budget of player 2 is \( b_2(p^*) = 1 \). Hence player 2 demands exactly the remaining quantity. Looking at player 1 we can see that, since she did not clinch anything up to \( p^* \), her remaining budget is equal to her original budget, which was \( b_1 = 1 \). Thus the demand of player 1 at \( p^* \) is also \( 1/p^* \), again exactly equal to the remaining quantity. Therefore at \( p^* \) we have switched to a situation very similar to the second special case from above: both players have remaining budgets that are equal to 1, and at an initial price \( p^* \) simultaneously demand exactly the available quantity. Thus, the calculations of the second special case of above, setting \( Q = 1/p^* \), describe the course of the auction from this point until the end. In other words, we see that the general construction is simply a combination of the two special cases studied above. Note that the course of the above auction stops whenever the price reaches the point \( \min(v_1, v_2) \), and this can be in any of the three parts of the auction – at \( p < 1 \), at \( 1 < p \leq p^* \), or at \( p > p^* \). This description gives us the general mechanism:

**Definition 7.6 (General Mechanism)** Assume \( b_1 = 1 \leq b_2 \) and initial quantity of 1. Let \( p^* = e^{b_2 - 1} \).

- If \( \min(v_1, v_2) < p^* \) then run Mechanism A.
- Otherwise, allocate to player 2 an initial quantity of \( 1 - 1/p^* \) for a total price \( b_2 - 1 \). Allocate the remaining quantity \( Q = 1/p^* \) using Mechanism B, where the initial budget of player 2 at the mechanism is \( b_2 = 1 \), and the rest of the parameters are unchanged.

**Proposition 7.7** The General Mechanism is Pareto-optimal and truthful in the case of publicly known budgets.

**Proof:** We first prove Pareto-optimality. If \( \min(v_1, v_2) < p^* \) then the outcome is determined by mechanism A, hence is Pareto-optimal by proposition 7.2. If \( \min(v_1, v_2) \geq p^* \), then mechanism B is run, and inside it we always enter the second option, which implies that the high-value player pays 1. If this is player 1 then this exhausts her budget, and if this is player 2 then her total payment is \( (b_2 - 1) + 1 = b_2 \), so her budget exhausted as well. Thus by proposition 2.4 the outcome is indeed Pareto-optimal.

We now prove truthfulness. Consider first the incentives of player 1. If \( v_2 < p^* \) then mechanism A is used, no matter what player 1 reports, and the claim follows from proposition 7.2. Otherwise
\[ v_2 > p^*. \] If \( v_1 < p^* \) then by the properties of mechanism B player 1 prefers receiving zero utility to receiving some quantity as a result of declaring some \( z > p^* \), since, in mechanism B, when \( v_1 < p^* \) player 1 gets nothing. Thus in this case player 1 maximizes utility by the truthful declaration. If \( v_1 > p^* \) then if she declares some \( z < p^* \) she gets zero utility while if she declares \( v_1 \) she gets a non-negative utility since mechanism B is individually rational. Thus she prefers to declare some \( z > p^* \) and since mechanism B is truthful it must be that \( z = v_1 \). This proves truthfulness for player 1.

Now consider player 2. If \( v_1 < p^* \) then the proof is a before. Otherwise \( v_1 > p^* \). If \( v_2 < p^* \) then player 2 prefers getting nothing from mechanism B to getting some positive quantity as a result of declaring some \( z > p^* \), and she prefers getting from mechanism A a quantity that results from declaring \( v_2 \) to getting \( 1 - 1/p^* \) and paying \( b_2 - 1 \) (which results from declaring \( z = p^* \)). Thus player 2 prefers to declare \( v_2 \) over declaring some \( z > p^* \), and therefore by the truthfulness of mechanism A she prefers to declare \( v_2 \) over any other declaration \( z \). If \( v_2 > p^* \) then player 2 prefers getting some quantity from mechanism B according to the declaration \( z = v_2 \) over not getting anything from mechanism B, since mechanism B is individually rational. Additionally, player 2 prefers the outcome \( x_2 = 1 - 1/p^*, p_2 = b_2 - 1 \) over any other outcome that results from mechanism A by declaring some \( z < p^* \), since \( v_2(1 - 1/p^*) - \ln p^* > v_2(1 - 1/z) - \ln z \). Thus player 2 prefers the outcome resulting from declaring \( v_2 \) over any other outcome that results from declaring some \( z < p^* \). By the truthfulness of mechanism B, declaring \( v_2 \) will maximize player 2’s utility. Therefore truthfulness for player 2 follows.

Individual-rationality follows from truthfulness, since a player can always obtain a zero utility by declaring \( v_i = 0 \). □

7.2 Uniqueness for equal (and known) budgets

To show uniqueness we cannot simply use similar arguments to the ones of the discrete case, since there we used induction on the number of items, while here the number of items is fixed, in some sense. Thus we use completely different arguments, and rely on the additional property of anonymity. As defined, mechanism B is not really anonymous, breaking the tie \( v_1 = v_2 \) “in favor” of \( v_2 \). An anonymous mechanism with the same properties can be obtained by “splitting” in case of a tie:

**Definition 7.8 (Mechanism C)**

- If \( v_1 = v_2 = v \leq 1 \) then \( x_1 = x_2 = 1/2 \) and \( p_1 = p_2 = v/2 \).
- If \( v_1 = v_2 = v > 1 \) then \( x_1 = x_2 = 1/2 \) and \( p_1 = p_2 = 1 - 1/(2v) \).
- If \( v_1 \neq v_2 \) then run mechanism B.

It is not hard to verify that mechanism C maintains the truthfulness and the Pareto-optimality of mechanism B. Moreover, we show:

**Theorem 7.9** Mechanism C is the only anonymous mechanism for the divisible good setting that satisfies truthfulness and Pareto-optimality.

**Proof:** Let us fix a mechanism that satisfies the above properties and reason about it. In the rest of the proof we denote the smaller value by \( v_i \), thus \( v_i \leq v_j \).
Step 1: We first handle the case of $v_i \leq 1$. If also $v_j < 1$ then $p_j \leq v_j < 1$ and thus Pareto-optimality implies $x_i = 0$ and $x_j = 1$. By the usual arguments of truthfulness we must have $p_j = v_i$. Now for values $v_j \geq 1$, if $x_j = 1$ then by truthfulness $p_j$ is determined by $x_j$ and thus is $p_j = v_i$. Otherwise $x_j > 0$ and thus by Pareto-optimality $p_j = 1$ but this is a contradiction to truthfulness since declaring a value $v_i < v_j'$ < 1 both increases $x_j$ and decreases $p_j$.

Step 2: We will now show that there exist functions $q(t)$ and $p(t)$ such that whenever $v_i < v_j$ then $x_i = q(v_i)$, $p_i = p(v_i)$, and $x_j = 1 - q(v_i)$, $p_j = 1$. I.e. the low player’s value determines the allocation between the two players as well as his own payment, while the high player exhausts his values of $p$. Optimality implies $p_j < t < v_j$ such that whenever $1 < v_i < v_j$, then it also only depends on $V_i$, and thus by Pareto-optimality, $x_i = 1 - x_j$ and thus it also only depends on $V_i$. But then a bidder with $p_j < v_j' < 1 < v_i$ that, according to step 1, gets nothing, would be better off declaring $v_j$ and getting positive utility, in contradiction to truthfulness. Thus $p_j = 1$ whenever $1 < v_i < v_j$. Thus, by truthfulness, for a fixed $v_i$, different values of $v_j$ must get the same $x_j$, i.e. $x_j$ depends only on $v_i$. By Pareto-optimality, $x_i = 1 - x_j$ and thus also only depends on $V_i$, and then by truthfulness $p_i$ must be determined uniquely by $x_i$ and thus depends only on $v_i$.

Step 3: Using truthfulness as usual, we have that for any $1 < t < t' < v_j$: $q(t') - q(t) \leq t(q'(t') - q(t))$. As usual this implies that $dp/dt = t \cdot dq/dt$ or, more precisely, since we do not know that $q$ is differentiable or even continuous, that $p(t) = t q(t) - \int_1^t q(x) dx$, where integrability of $q$ is a direct corollary of its monotonicity. (This already takes into account the boundary condition that for $t$ approaching 1 from above, $q(x)$ must approach 0, as otherwise for the fixed limit $\delta > 0$ we will have that for every value of $v_2 > v_1 > 1$, we will have $x_2 \leq 1 - \delta$, which by IR implies $p_2 < 1$ and thus contradicts PO.)

Step 4: Using truthfulness we have that for $1 < t < v_j < t'$: $q(t') - p(t) \geq t(1 - q(t')) - 1$ and $\int_1^t q(x) dx \geq \int_1^{1 - q(v_j)} (1 - q(v_j)) - 1$. Letting $t, t'$ approach $v_j$ we have that $t q(t) - p(t) = t(1 - q(t)) - 1$, i.e. $p(t) = 1 + t(2q(t) - 1)$ for all $t$ except for at the at most countably many points of discontinuity of $q$.

Step 5: Combining the last two steps we have $1 + t(2q(t) - 1) = t q(t) - \int_1^t q(x) dx$, i.e. $q(t) = 1 - t/(1 - \int_1^t q(x) dx)$. This solution is a direct corollary of its monotonicity of $q$. The solution to this differential equation, is $q(t) = 1/2 - 1/(2t^2)$ which gives $p(t) = 1 - 1/t$. The uniqueness of solution is implied since if another function satisfies the equation everywhere except for countably many points, then the difference function $d(t)$ would satisfy $d(t) = -(\int_1^t d(x) dx)/t$ everywhere except for countably many points, which only holds for $d(t) = 0$. □

7.3 The impossibility for private budgets

From theorem 7.9 we rather easily deduce:

Theorem 7.10 There exists no anonymous, truthful, and Pareto-optimal mechanism for the divisible good setting, for the case of privately known budgets $b_1, b_2$.

Proof: We first note that by direct scaling of theorem 7.9 we have that the only mechanism that satisfies all requirements of the claim for the case of a publicly known budget $b_1 = b_2 = B$ gives $x_i = (1 - B^2/v_i^2)/2$, $p_i = B(1 - B/v_i)$, $x_j = (1 + B^2/v_j^2)/2$, $p_j = 1$ for the case $1 < v_i < v_j$, and $x_j = 1$, $p_j = v_i$, $x_i = 0$, $p_i = 0$ for the case $v_i < 1$ and $v_i < v_j$. 23
Let us now assume to the contrary that an auction that satisfies all requirements of the claim exists, then for any fixed values of $b_1, b_2$ it must be identical to the scaled version of mechanism C. Now let us look at a few cases with $v_1 = 2, v_2 = 2 + \epsilon$. First let us look at the case $b_1 = b_2 = 1$. The previous theorem mandates that in this case $x_1 = 3/8, p_1 = 1/2$ and $x_2 = 5/8, p_2 = 1$, (and thus $u_2 = 1/4 + O(\epsilon)$.)

Now let us look at the case where $b_1 = b_2 = 2 - \epsilon$. Again the theorem 7.9 with scaling mandates that $x_1 > 0$ and also $u_1 > 0$.

Now let us look at the case of $b_1 = 1$ and $b_2 = 2 - \epsilon$. If $x_2 < 1$ then, by PO, $p_2 = b_2 = 2 - \epsilon$, and thus $u_2 < 2\epsilon$, which means that player 2 has a profitable lie stating $b_2 = 1$. Thus $x_2 = 1$ and $x_1 = 0$, but then player 1 has a profitable lie stating that $b_1 = 2 - \epsilon$.

\[\square\]

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References


We need and

Property (b) implies that there exists an index $1 \leq i \leq n$ such that, for any index $i \leq k$, $x_i > 0$ and $p_i = b_j$, for any index $i > k$, $x_i = 0$, and at $k$ itself, $x_k > 0$. Let $\Delta = \sum_{i=1}^{k-1} (x_i - x'_i)$. For any $i$ we need $u'_i \geq u_i$, which implies $p'_i - p_i \leq v_i \cdot (x'_i - x_i)$. We make several observations. First,

$$\sum_{i=k}^{n} (p'_i - p_i) \leq v_k(x'_k - x_k) + \sum_{i=k+1}^{n} v_i(x'_i - x_i) \leq v_k \sum_{i=k}^{n} (x'_i - x_i) = \Delta \cdot v_k$$

A Proof of Claim 2.4

Recall that we need to show that an outcome $\{(x_i, p_i)\}$ is Pareto-optimal in the infinitely divisible case if and only if (a) $\sum_i x_i = 1$ and (b) for all $i$ such that $x_i > 0$ we have that for all $j$ with $v_j > v_i$, $p_j = b_j$.

We first show that if either (a) or (b) do not hold then the outcome is not Pareto. If $\sum_i x_i < 1$ we simply add an additional quantity to some player for no additional charge, thus making him strictly better off while not harming any other player. Otherwise $\sum_i x_i = 1$ and there exists a player $i$ with $x_i > 0$ and a player $j$ with $v_j > v_i$ and $p_j < b_j$. Fix some $\epsilon$ such that $\epsilon \cdot v_i < b_j - p_j$. Construct an outcome $(x', p')$ such that $x'_i = x_i - \epsilon$, $x'_j = x_j + \epsilon$, $p'_i = p_i - \epsilon \cdot v_i$, and $p'_j = p_j - \epsilon \cdot v_i$, all other players get the same quantity and pay the same price. Notice that $\sum_i p'_i = \sum_i p_i$ and that $(x', p')$ is indeed a valid outcome. It is straight-forward to verify that $i$’s utility remains the same while $j$’s utility strictly increases.

For the other direction, fix an outcome $(x, p)$ that satisfies (a) and (b). We will show that any other outcome $(x', p')$ cannot be a Pareto improvement to $(x, p)$ (as in Def. 2.3), implying that $(x, p)$ is Pareto. Since (a) holds then $\sum_i x_i = 1$. Rename the players such that $v_1 \geq v_2 \geq \cdots \geq v_n$. Property (b) implies that there exists an index $1 \leq k \leq n$ such that, for any index $i < k$, $x_i > 0$ and $p_i = b_j$, for any index $i > k$, $x_i = 0$, and at $k$ itself, $x_k > 0$. Let $\Delta = \sum_{i=1}^{k-1} (x_i - x'_i)$. For any $i$ we need $u'_i \geq u_i$, which implies $p'_i - p_i \leq v_i \cdot (x'_i - x_i)$. We make several observations. First,
where the second inequality follows since $x_i = 0$ for any $i > k$, and the third inequality follows since $\sum_{i=1}^{k-1} (x_i - x'_i) - \sum_{i=k}^{n} (x'_i - x_i) = 0$. Second,

\[
\sum_{i=1}^{k-1} (p_i - p'_i) \geq \sum_{1 \leq i \leq k-1 : x_i \geq x'_i} (p_i - p'_i) \geq \sum_{1 \leq i \leq k-1 : x_i \geq x'_i} (x_i - x'_i) v_i \geq \sum_{1 \leq i \leq k-1 : x_i \geq x'_i} (x_i - x'_i) v_k \geq \Delta \cdot v_k
\]

where the first inequality follows since $p_i = b_i \geq p'_i$ for any $i < k$. Now, if there exists $1 \leq i \leq k-1$ such that $x_i < x'_i$ then the above argument yields $\sum_{i=1}^{k-1} (p_i - p'_i) > \Delta \cdot v_k$. We then get $\sum_{i=1}^{k-1} (p_i - p'_i) - \sum_{i=k}^{n} (p'_i - p_i) > 0$. In other words, $\sum_i p_i > \sum_i p'_i$, a contradiction to the definition of a Pareto improvement. Therefore assume that $x_i \geq x'_i$ for any $1 \leq i \leq k-1$. This implies that

\[
\sum_{i=1}^{k-1} (x_i - x'_i) v_i \geq \Delta \cdot v_k \geq \sum_{i=k}^{n} (x'_i - x_i) v_i.
\]

Putting together these four inequalities, we get

\[
\sum_{i} (u_i - u'_i) = \sum_{i=1}^{k-1} (p_i - p'_i) - \sum_{i=k}^{n} (p'_i - p_i) + \sum_{i=1}^{k-1} (x_i - x'_i) v_i - \sum_{i=k}^{n} (x'_i - x_i) v_i \geq 0.
\]

As a result, $u_i = u'_i$ for any player $i$, hence $(x', p')$ is not a Pareto improvement for $(x, p)$ since there does not exist a player $i$ with $u'_i > u_i$. This concludes the proof of the claim.